18 Duality for $SO(N)$

18.1 The $SO(N)$ Theories and Their Classical Moduli Spaces

	$SO(N)$	$SU(F)$	$U(1)_R$
Q	$\square$	$\square$	$\frac{F+2-N}{F}$

Recall that the adjoint of $SO(N)$ is the two-index antisymmetric tensor. For odd N, there is one spinor representation, while for even N there are two inequivalent spinors. For $N = 4k$ the spinors are self-conjugate, while for $N = 4k + 2$ the two spinors are complex conjugates. Since there are no dynamical spinors in our theory, static spinor sources cannot be screened, so there is a distinction between confining and Higgs phases.

$SO(2N+1)$		
Irrep	$d(r)$	$2T(r)$
$\square$	$2N+1$	2
S	2^N	2^{N-2}
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$N(2N+1)$	$4N-2$
$\square\square$	$(N+1)(2N+1)-1$	$4N+6$

$SO(2N)$		
Irrep	$d(r)$	$2T(r)$
$\square$	$2N$	2
S	2^{N-1}	2^{N-3}
$\bar{S}, (S')$	2^{N-1}	2^{N-3}
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$N(2N-1)$	$4N-4$
$\square\square$	$N(2N+1)-1$	$4N+4$

The one-loop β function coefficient for $N > 4$ is

$$b = 3(N-2) - F \tag{18.1}$$

Solving the D-flatness conditions one finds that up to flavor transformations, we classical vacua for $F < N$ are given by

$$\langle \Phi \rangle = \begin{pmatrix} \overline{v}_1 & & \\ & \ddots & \\ & & \overline{v}_F \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \tag{18.2}$$

At a generic point in the classical moduli space the $SO(N)$ gauge symmetry is broken broken to $SO(N-F)$ and there are $NF - N(N-1) + (N-F)(N-F-1)$ massless chiral supermultiplets. For $F \geq N$ the vacua are:

$$\langle \Phi \rangle = \begin{pmatrix} v_1 & & & 0 & \dots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & v_N & 0 & \dots & 0 \end{pmatrix} \tag{18.3}$$

At a generic point in the moduli space the $SO(N)$ gauge symmetry is broken completely and there are $NF - N(N-1)$ massless chiral supermultiplets. We can descibe these light degrees of freedom in a gauge invariant way by scalar "meson" and (for $F \geq N$) "baryon" fields and there superpartners:

$$M_{ji} = \Phi_j \Phi_i \tag{18.4}$$

$$B_{[i_1,\ldots,i_N]} = \Phi_{[i_1} \ldots \Phi_{i_N]} \tag{18.5}$$

where [] denotes antisymmetrization.

Up to flavor transformations the moduli space is described by:

$$\langle M \rangle = \begin{pmatrix} v_1^2 & & & & & \\ & \ddots & & & & \\ & & v_N^2 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \tag{18.6}$$

$$\langle B_{1,\ldots,N} \rangle = v_1 \ldots v_N \tag{18.7}$$

$$\tag{18.8}$$

with all other components set to zero. The rank of M is at most N. If the rank of M is N, then $B = \pm\sqrt{\det{}' M}$.

18.2 The Dynamical Superpotential for $F < N - 2$

To construct the effective superpotential should look at how the chiral superfields transform under the anomalous axial $U(1)_A$.

$$\begin{array}{ccc} & U(1)_A & U(1)_R \\ W^a & 0 & 1 \\ \Lambda^b & 2F & 0 \\ \det M & 2F & 2(F+2-N) \end{array} \tag{18.9}$$

So we see it is possible to generate a dynamical superpotential

$$W_{\rm dyn} = c_{N,F} \left(\frac{\Lambda^b}{\det M} \right)^{\frac{1}{N-2-F}} . \tag{18.10}$$

for $F < N - 2$.

18.3 Duality

For $F \geq 3(N-2)$ we lose asymptotic freedom, so the theory can be understood as a weakly coupled low-energy effective theory. For F just below $3N$ we have an infrared fixed point.

A solution to the anomaly matching for $F > N - 2$ is given by:

	$SO(F-N+4)$	$SU(F)$	$U(1)_R$
q	$\square$	$\overline{\square}$	$\frac{N-2}{F}$
M	$\mathbf{1}$	$\square\square$	$\frac{2(F+2-N)}{F}$

For $N > N - 3$ this theory admits a unique superpotential:

$$W = \frac{M_{ji}}{2\mu} \phi^j \phi^i \tag{18.11}$$

The dual theory also has baryon operators:

$$\widetilde{B}^{[i_1,\ldots,i_{\widetilde{N}}]} = \phi^{[i_1} \ldots \phi^{i_{\widetilde{N}}]} \tag{18.12}$$

There are additional hybrid "baryon" operators in both theories since the adjoint is an antisymmetric tensor. In the original $SO(N)$ theory we have:

$$\begin{aligned} h_{[i_1,\ldots,i_{N-4}]} &= W_\alpha^2 \Phi_{[i_1} \ldots \Phi_{i_{N-4}]} \\ H_{[i_1,\ldots,i_{N-2}]\alpha} &= W_\alpha \Phi_{[i_1} \ldots \Phi_{i_{N-4}]} \end{aligned} \tag{18.13}$$

While in the dual theory we have:

$$\begin{aligned}
\widetilde{h}^{[i_1,\ldots,i_{\widetilde{N}-4}]} &= \widetilde{W}_\alpha^2 \phi^{[i_1}\ldots\phi^{i_{\widetilde{N}-4}]} \\
\widetilde{H}_\alpha^{[i_1,\ldots,i_{\widetilde{N}-2}]} &= \widetilde{W}_\alpha \phi^{[i_1}\ldots\phi^{i_{\widetilde{N}-4}]}
\end{aligned} \tag{18.14}$$

The two theories have a mapping

$$\begin{aligned}
&M \leftrightarrow M \\
&B_{i_1,\ldots,i_N} \leftrightarrow \epsilon_{i_1,\ldots,i_F}\widetilde{h}^{i_1,\ldots,i_{\widetilde{N}-4}} \\
&h_{i_1,\ldots,i_{N-4}} \leftrightarrow \epsilon_{i_1,\ldots,i_F}\widetilde{B}^{i_1,\ldots,i_{\widetilde{N}}} \\
&H_\alpha^{[i_1,\ldots,i_{N-2}]} \leftrightarrow \epsilon_{i_1,\ldots,i_F}\widetilde{H}_\alpha^{[i_1,\ldots,i_{\widetilde{N}-2}]}
\end{aligned} \tag{18.15}$$

The dual β function is

$$\beta(\widetilde{g}) \propto -\widetilde{g}^3(3(\widetilde{N}-2)-F) = -\widetilde{g}^3(2F-3(N-2)) \tag{18.16}$$

So the dual theory loses asymptotic freedom when $F \leq 3(N-2)/2$. When

$$F = 3\widetilde{N} - \epsilon\widetilde{N} \tag{18.17}$$

there is a perturbative fixed point. One can check that the exact β function vanishes in this range using the relation between dimensions and R charges in a superconformal theory. So we have found that $SO(N)$ with F vectors has an interacting IR fixed point for $3(N-2)/2 < F < 3(N-2)$.

For $N-2 \leq F \leq 3(N-2)/2$ the IR fixed point of the dual theory is trivial and we find in the IR free massless composite gauge bosons, quarks, mesons, and their superpartners.

One can check that adding mass term in the original theory and a linear meson term in the dual theory leads to the correct reduction of flavors and dual colors.

18.4 Some Special Cases

For $F \leq N-5$, $SO(N)$ breaks to $SO(N-F) \supset SO(5)$, which undergoes gaugino condensation and produces the dynamical superpotential:

$$W_{\rm dyn} \propto \langle\lambda\lambda\rangle \propto \left(\frac{16\Lambda^{3(N-2)-F}}{\det M}\right)^{\frac{1}{N-2-F}} . \tag{18.18}$$

For $F = N - 4$, $SO(N)$ breaks to $SO(4) \sim SU(2)_L \times SU(2)_R$, so there are two gaugino condensates

$$W_{\rm dyn} = 2\langle\lambda\lambda\rangle_L + 2\langle\lambda\lambda\rangle_R = \frac{1}{2}(\epsilon_L + \epsilon_R)\left(\frac{16\Lambda^{2N-1}}{\det M}\right)^{\frac{1}{2}} . \qquad (18.19)$$

where

$$\epsilon_s = \pm 1 \qquad (18.20)$$

So there are four vacua corresponding to two physically distinct branches: one with $(\epsilon_L + \epsilon_R) = \pm 2$ and the other with $(\epsilon_L + \epsilon_R) = 0$. The first branch has runaway vacua, while the second has a quantum moduli space. At $M = 0$, the composite M satisfies the 't Hooft anomaly matching conditions. One can check that this only happens for $F = N - 4$. So we have another example of confinement without chiral symmetry breaking, this time without any baryons. Integrating out a flavor on the first branch gives the correct runaway vacua of the previous case, while on the second branch we find no supersymmetric vacua, which is a consistency check.

For $F = N-3$, $SO(N)$ breaks to $SO(4) \sim SU(2)_L \times SU(2)_R$, which then breaks to $SU(2)_d \sim SO(3)$ so there are instanton effects (since $\Pi_3(G/H) = \Pi_3(SU(2)) = Z$) and gaugino condensation

$$W_{\rm dyn} = 4(1+\epsilon)\frac{\Lambda^{2N-3}}{\det M} . \qquad (18.21)$$

where

$$\epsilon = \pm 1 \qquad (18.22)$$

corresponding to the two phases of the gaugino condensate. So there are two physically distinct branches: one with $\epsilon = 1$ and the other with $\epsilon = -1$. The first has runaway vacua, while the second has a quantum moduli space. Integrating out a flavor, we would need to find two branches again, so $W \neq 0$ even on the second branch. For this to be true we must have some other fields that interact with M. We also know that M does not match the anomalies by itself. The solution of the anomaly matching is given by:

	$SU(F)$	$U(1)_R$
q	$\overline{\square}$	$\frac{N-2}{F}$
M	$\square\square$	$\frac{2(F+2-N)}{F}$

The most general superpotential is:

$$W = \frac{1}{2\mu} Mqqf\left(\frac{\det M\, Mqq}{\Lambda^{2N-2}}\right) \tag{18.23}$$

Adding a mass term gives

$$q_F = \pm iv \tag{18.24}$$

which gives us the correct number of ground states. Note that the operator mapping must be:

$$q \leftrightarrow h = Q^{N-4} W_\alpha W^\alpha \tag{18.25}$$

which is a hybrid operator. For $N = 4$ this is a gluinoball. This is an example of confinement without chiral symmetry breaking with hybrids.

Starting with the $F = N$ dual which has an $SO(4)$ gauge group, and integrating out a flavor there will be instanton effects when we break to $SO(3)$ so the dual superpotential is modified in the case $F = N - 1$:

$$W = \frac{M_{ji}}{2\mu}\phi^j\phi^i - \frac{1}{64\Lambda^{2N-5}}\det M \tag{18.26}$$

For $F = N - 2$ we see in the original theory that we can generically break to $SO(2)$, while in the dual we also break so $SO(2) \sim U(1)$. This case needs a seperate treatment.

References

[1] "Lectures on supersymmetric gauge theories and electric-magnetic duality," by K. Intriligator and N. Seiberg, hep-th/9509066.

[2] K. Intriligator and N. Seiberg, "Duality, monopoles, dyons, confinement and oblique confinement in supersymmetric SO(N(c)) gauge theories," Nucl. Phys. **B444** (1995) 125 hep-th/9503179.